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Enhanced Structural Damage Detection Using Alternating Projection Methods

Musa O. Abdalla,* Karolos M. Grigoriadis,† and David C. Zimmerman‡
University of Houston, Houston, Texas 77204-4792

Alternating projection algorithms are examined for the solution of damage detection problems in structures. The damage detection problem is formulated as a feasibility problem to find a damaged stiffness matrix that is close to the refined stiffness matrix of the undamaged structure and that satisfies the necessary symmetry, sparsity, positive definiteness, eigenequation, and damage localization constraints. Alternating projection methods are proposed to utilize the orthogonal projections onto these constraint sets in an iterative fashion to find a solution that best satisfies these constraints. In addition, directional alternating projections that exploit the geometry of the damage detection feasibility problem are introduced to improve the computational efficiency of the approach. The techniques are applied to detect damage in a simulated cantilever beam model and in the NASA eight-bay truss damage detection experimental test bed.

Nomenclature

$\|A\|$	= Frobenius norm of the matrix A
A^T	= transpose of the matrix A
A^+	= Moore–Penrose generalized inverse of the matrix A
$\langle A, B\rangle$	= inner product of two matrices A and B
$A \circ B$	= Hadamard matrix product of two matrices A and B
K	= analytical (healthy) stiffness matrix
K_d	= damaged stiffness matrix
M	= analytic mass matrix
$\mathcal{P}_C(X_0)$	= projection of X_0 onto C
$\boldsymbol{q}$	= generalized coordinates vector
V_d	= measured mode-shape matrix
$\boldsymbol{v}_{di}$	= measured mode shapes
Ω_d	= measured eigenfrequency matrix
ω_{di}	= measured eigenfrequencies

I. Introduction

TECHNIQUES for damage detection and health monitoring of aerospace, civil, and mechanical engineering structures are essential in determining their safety, reliability, and operational life. For example, regular monitoring and assessment of the structural integrity of the Space Station Freedom would be necessary to estimate potential damage by micrometeoroids and orbital debris impact, Space Shuttle dockings, and fatigue due to nominal loading or accidents. The structural health monitoring problem consists of obtaining information about the existence, location, and extent of damage in structures using nondestructive methods. Based on experimental modal analysis and signal processing techniques, a promising approach for health monitoring of structures is to monitor and interpret changes on structural dynamic measurements. The extraction of natural frequency and mode shape information of a vibrating structure can be accomplished using modern vibration testing equipment and instrumentation. Modal and structural dynamic data of a vibrating structure can be utilized for cost-effective health monitoring and operational life assessment information without a need for dismantling the structure.

Most prior work on damage detection of structures is directed toward the general framework of finite element model (FEM) refinement methods. The motivation behind these techniques is the need to refine and validate FEM structural models before they are acceptable as accurate models of the structure.[1] Model updating, model correction, and health monitoring approaches have attracted substantial attention in the past few years, as evident in several recent review articles.[2–5] Optimal matrix update methods that seek to determine the system property matrices, such as the stiffness matrix, using measured test data are used extensively in FEM refinement and damage detection. Health monitoring information can be associated with changes in the system matrices; for example, stiffness reduction can be attributed to damage. In their work on optimal orthogonalization of measured modes, Baruch and Bar Itzhack[1] obtained a closed-form solution for the minimum Frobenius norm adjustment to the mass-weighted structural stiffness matrix that incorporates the measured frequencies and mode shapes. However, the zero/nonzero sparsity pattern of the original stiffness matrix may be destroyed. Algorithms by Kabe,[6] Kammer,[7] Smith and Beattie,[8] and Smith[9] were developed recently to preserve the original stiffness matrix pattern, thereby, preserving the original load paths of the structural model. Although these methods appear to provide satisfactory results for model refinement, their use for damage detection is questionable inasmuch as damage typically results in localized changes in the system property matrices. In addition to the cited optimal matrix update methods, control-based eigenstructure assignment techniques were recently proposed to determine the pseudocontrol that would be required to produce the measured modal properties.[10] Also, a minimum rank perturbation theory (MRPT) approach has been developed by Zimmerman and Kaouk[11,12] that exploits the finite element representation of structural damage.

In the present work, alternating projection algorithms are proposed to provide improved optimal matrix updates for damage detection purposes. Alternating projections have been used in the past to solve statistical estimation, image reconstruction, and control design problems.[13–16] Also, some recent iterative optimal matrix update methods for damage detection are essentially alternating projections although the proper justification and interpretation of the methods has not been examined there.[9,10] In the present work, we provide the theoretical foundations and formulation of the alternating projection method for damage detection and a rigorous justification of their convergence and properties. Also, an improved directional alternating projection damage detection algorithm is introduced, which results in enhanced convergence. Finally, two computational examples are provided. We use the proposed methods to detect damage in a four-degree-of-freedom (DOF) vibrating bar model and in the NASA eight-bay truss damage detection test bed. These examples demonstrate the capabilities of the proposed alternating projection algorithms to detect localized structural damage.

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*Graduate Research Assistant, Department of Mechanical Engineering.

†Assistant Professor, Department of Mechanical Engineering. Senior Member AIAA.

‡Associate Professor, Department of Mechanical Engineering.

The notation to be used in this paper is as follows: A^+ denotes the Moore–Penrose generalized inverse of a matrix A, that is, if

$$A = [U_1 \quad U_2]\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_1^T\\ V_2^T\end{bmatrix}$$

is the singular value decomposition of A, then

$$A^+ = V_1\Sigma_1^{-1}U_1^T$$

The symbol $\circ$ is to denote element-by-element multiplication of two matrices (Hadamard matrix product). The Frobenius norm of a matrix A is defined as

$$\|A\| = \left[\sum_{i,j}A_{ij}^2\right]^{\frac{1}{2}} = [\mathrm{tr}(AA^T)]^{\frac{1}{2}}$$

and the inner product of two matrices A and B as

$$\langle A, B\rangle = \mathrm{tr}(AB^T)$$

where A and B have the same dimensions. The standard notation $>, \geq$ ($<, \leq$) is used to denote the positive (negative) definite and semidefinite ordering of matrices as well as standard set notation.

II. Problem Formulation

Consider an undamped n-DOF finite element model of an undamaged structure represented by the general equation of motion

$$M\ddot{\boldsymbol{q}} + K\boldsymbol{q} = 0 \tag{1}$$

where M and K are the $n\times n$ analytical mass and stiffness matrices, respectively, and $\boldsymbol{q}$ is an $n\times 1$ vector of generalized coordinates (displacements at the nodes). We assume that the FEM (1) is a refined model of the structure, that is, the measured and analytical modal properties are in agreement (possible through the use of a model refinement procedure). The damage detection problem consists of using measured mode shapes and natural frequencies to compute the structural stiffness matrix and to make an assessment of the location and extent of damage based on stiffness reduction at individual structural elements.

The eigenproblem of Eq. (1) is given by

$$\left(-\omega_i^2M + K\right)\boldsymbol{v}_i = 0, \qquad i = 1,\ldots,n \tag{2}$$

where ω_i and $\boldsymbol{v}_i$ denote the ith natural frequency and the ith mode shape, respectively, of the healthy structure. These mode shapes are chosen to be mass normalized, that is, $\boldsymbol{v}_i^TM\boldsymbol{v}_i = 1$.

Suppose that a postdamage modal testing provides a set of eigenfrequencies and eigenvectors ω_{di} and $\boldsymbol{v}_{di}$, $i = 1,\ldots,p$, where $p \ll n$. In this work, it is assumed that the dimension of the measured eigenvectors is the same as the analytical eigenvectors. The postdamage eigenvalue problem is now

$$\left(-\omega_{di}^2M + K_d\right)\boldsymbol{v}_{di} = 0, \qquad i = 1,\ldots,n \tag{3}$$

where K_d is the stiffness matrix of the damaged structure. We have assumed that the effect of damage on the mass properties of the structure is negligible. The set of eigenequations (3) can be written in a matrix form as follows:

$$K_dV_d = MV_d\Omega_d^2 \tag{4}$$

where $\Omega_d = \mathrm{diag}(\omega_{d1}, \omega_{d2},\ldots,\omega_{dp})$ is the $p\times p$ diagonal matrix of measured eigenfrequencies and $V_d = [\boldsymbol{v}_{d1}, \boldsymbol{v}_{d2},\ldots,\boldsymbol{v}_{dp}]$ is the $n\times p$ mass-orthogonal modal matrix. The damage detection problem reduces to identifying the damaged stiffness matrix K_d from the eigenmeasurements ω_{di} and $\boldsymbol{v}_{di}$.

Given the measured eigenfrequency matrix Ω_d and the measured mode-shape matrix V_d, the optimal matrix update of Baruch and Bar Itzhack[1] provides the optimally modified stiffness matrix K_d^* that solves the following minimization problem:

$$\underset{K_d^*}{\mathrm{minimize}}\left\|M^{-\frac{1}{2}}\left(K_d^* - K\right)M^{-\frac{1}{2}}\right\| \tag{5}$$

subject to

$$K_d^*V_d = MV_d\Omega_d^2, \qquad K_d^* = K_d^{*T} \tag{6}$$

where $\|\cdot\|$ denotes the Frobenius norm of a matrix. A closed-form solution to the preceding minimization problem is given by

$$K_d^* = K - KXX^TM - MXX^TK + MXX^TKXX^TM + MX\Omega_d^2X^TM \tag{7}$$

$$X = V_d\left(V_d^TMV_d\right)^{-\frac{1}{2}} \tag{8}$$

Hence, the Baruch–Bar Itzhack formula (7) provides the closest matrix update that is consistent with the measured modal data. Although the positive definiteness of the stiffness matrix is guaranteed in Eq. (7), the structural connectivity as represented by the zero/nonzero pattern of the structural stiffness matrix is lost in the process.

To preserve structural connectivity and to obtain more accurate damage detection information, we seek to solve the following optimal matrix update problem:

$$\underset{K_d^*}{\mathrm{minimize}}\left\|K_d^* - K\right\| \tag{9}$$

subject to the eigenequation constraint (4) and a sparsity constraint on the solution matrix K_d^*. We define an $n\times n$ structural stiffness matrix K_d to be K-sparse if K_d has the zero/nonzero pattern of the analytical stiffness matrix K. Hence, a K-sparse matrix preserves the FEM connectivity of the analytical model. Let us define the following sets that correspond to the physical and eigenequation requirements on K_d:

$$\begin{aligned}
\mathcal{C}_1 &= \{K_d\colon K_d \geq 0\}\\
\mathcal{C}_2 &= \left\{K_d\colon K_d = K_d^T\right\}\\
\mathcal{C}_3 &= \{K_d\colon K_d \text{ is } K\text{-sparse}\}\\
\mathcal{C}_4 &= \left\{K_d\colon K_dV_d = MV_d\Omega_d^2\right\}
\end{aligned} \tag{10}$$

Then the optimal matrix update problem we seek to solve is the following:

$$\underset{K_d^*}{\mathrm{minimize}}\left\|K_d^* - K\right\| \tag{11}$$

subject to

$$K_d^* \in C_1\cap C_2\cap C_3\cap C_4 \tag{12}$$

where $\cap$ is the set intersection operation.

The presence of measurement noise in the modal data often results in an inconsistent family of constraint sets C_1, C_2, C_3, and C_4. That is, it is possible that no positive definite, symmetric, K-sparse matrix K_d^* exists that satisfies eigenequation (4), i.e., $C_1\cap C_2\cap C_3\cap C_4 = \emptyset$. Hence, in this case we are interested in finding the positive definite, symmetric, K-sparse matrix K_d^* that is nearest to the eigenequation constraint set C_4, i.e., we seek to solve the optimization problem

$$\underset{K_d^*}{\mathrm{minimize}}\left\|K_d^* - K_d\right\| \tag{13}$$

subject to

$$K_d^* \in C_1\cap C_2\cap C_3 \tag{14}$$

and

$$K_d \in C_4 \tag{15}$$

A significant difference of the damage detection problem from the model refinement problem is that localized damage results in localized changes in the structural stiffness matrix. To enhance the detection of localized damage in our approach, a masking procedure is proposed. The objective of masking is to ensure that the damage detection algorithm will focus on the damaged stiffness matrix elements that have changed significantly and will neglect small changes that can be attributed to noise. In addition, undesirable distribution of the identified stiffness matrix changes over all elements of the stiffness matrix will be avoided. To this end, small changes in the structural stiffness matrix K_d^*, e.g., less that 10% of the maximum change, are neglected. We will say that the matrix K_d^* is masked if such small changes in the stiffness matrix elements have been neglected. We define the following masking constraint set:

$$\mathcal{C}_3' = \{K_d \colon K_d \text{ is masked}\} \tag{16}$$

Notice that if K_d^* is a K-sparse matrix then the masked K_d^* matrix is also K-sparse, that is, $C_3 \subset C_3$, where $\subset$ denotes the subset relation between two sets. Hence, for the noisy damage detection problem the constraint C_3 in the optimization problems (11) and (12) and (13–15) should be replaced by $\mathcal{C}_3'$.

It is noted that all constraint sets $\mathcal{C}_i$ in Eq. (10) have a simple analytical representation, and they correspond to simple geometric constraints in the space of $n \times n$ matrices. In the following section, we present an alternating projection approach that exploits the geometry of these constraints to obtain a solution to the optimization problems (11) and (12) and (13–15).

III. Alternating Projections

The alternating projection approach utilizes the projections onto the constraint sets in an iterative fashion to obtain solutions of feasibility and optimization problems. These techniques have been used in the past to solve statistical estimation, image restoration, and control design problems.[13–16] In this paper we present the basic concepts of the alternating projection methods as they are applicable to our matrix optimization and feasibility problems for damage detection.

Let $\mathcal{M}$ denote the vector space of the $n \times n$ matrices equipped with the Frobenius norm

$$\|X\| = \left[\sum X_{ij}^2\right]^{\frac{1}{2}} = [\operatorname{tr}(XX^T)]^{\frac{1}{2}}$$

and the inner product

$$\langle X, Y\rangle = \operatorname{tr}(XY^T)$$

where $X, Y \in \mathcal{M}$. The damage detection problems we are interested in solving are described next.

The feasibility problem has the following form: Given the constraint sets $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ in $\mathcal{M}$, find a matrix X^* in the intersection of these sets, that is,

$$X^* \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \cdots \cap \mathcal{C}_n \tag{17}$$

The optimization problem has the following form: Given the constraint sets $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ in $\mathcal{M}$ and a matrix $X_0 \in \mathcal{M}$, find the matrix X^* in the intersection of these sets that is closest to X_0, i.e., solve the optimization problem

$$\underset{X^*}{\text{minimize}}\|X^* - X\| \text{ subject to } X^* \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \cdots \cap \mathcal{C}_k \tag{18}$$

The unfeasible optimization problem has the following form: Given two disjointed constraint sets $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathcal{M}$, find the matrix $X^* \in \mathcal{C}_1$ that is closest to the set $\mathcal{C}_2$, i.e., solve

$$\underset{X^*}{\text{minimize}}\|X^* - X\| \text{ subject to } X^* \in \mathcal{C}_1 \text{ and } X \in \mathcal{C}_2 \tag{19}$$

The feasibility problem (17) and the optimization problem (18) have a solution if and only if the constraint sets $\mathcal{C}_i$ have a nonempty intersection. The unfeasible optimization problem (19) considers the case where the two sets $\mathcal{C}_1$ and $\mathcal{C}_2$ do not intersect and seeks to find the matrix in $\mathcal{C}_1$ that is closest to the set $\mathcal{C}_2$.

The concept of convexity is important to characterize and determine a solution to the given matrix problems. Recall that a set $\mathcal{C}$ is convex if any two points (matrices in our case) that belong to the set define a line segment that is fully contained in the set. Convexity of the sets $\mathcal{C}_i$ guarantees that the optimization problem (18) and the unfeasible optimization problem (19) have unique solutions.[17]

Now consider a constraint set $\mathcal{C}$ in $\mathcal{M}$ and let X_0 be a given matrix in $\mathcal{M}$. We define the projection of X_0 onto $\mathcal{M}$ to be the matrix X^* in $\mathcal{C}$ that is closest to X, that is, X^* solves the optimization problem

$$\underset{X^*}{\text{minimize}}\left\|X^* - X_0\right\| \text{ subject to } X^* \in \mathcal{C} \tag{20}$$

The projection X^* of X_0 onto $\mathcal{C}$ is uniquely defined if the set $\mathcal{C}$ is convex, and this projection will be denoted by $\mathcal{P}_{\mathcal{C}}(X_0)$. In the sequel, we will present computational algorithms that utilize the projections onto the constraint sets $\mathcal{C}_i$ to solve the matrix feasibility and optimization problems (17–19). We will assume that the constraint sets $\mathcal{C}_i$ have simple geometry so that the projections $\mathcal{P}_{\mathcal{C}_i}$ can be computed efficiently.

A. Classic Alternating Projection Algorithm

The classic alternating projection algorithm[18,19] consists of sequential projections onto the constraint sets. Let X_0 be any given matrix in $\mathcal{M}$ and consider the following alternating projection sequence onto the constraint sets $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$:

$$\begin{aligned}
X_1 &= \mathcal{P}_{\mathcal{C}_1} X_0 \\
X_2 &= \mathcal{P}_{\mathcal{C}_2} X_1 \\
&\vdots \\
X_n &= \mathcal{P}_{\mathcal{C}_n} X_{n-1} \\
X_{n+1} &= \mathcal{P}_{\mathcal{C}_1} X_n \\
&\vdots \\
X_{2n} &= \mathcal{P}_{\mathcal{C}_n} X_{2n-1} \\
X_{2n+1} &= \mathcal{P}_{\mathcal{C}_1} X_{2n} \\
&\vdots \\
X_{3n} &= \mathcal{P}_{\mathcal{C}_n} X_{3n-1} \\
&\vdots
\end{aligned} \tag{21}$$

If the constraint sets $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ have a nonempty intersection, that is, if $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \cdots \cap \mathcal{C}_n \neq \emptyset$, then the sequence of matrices X_i converges to a matrix X^* in the intersection $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \cdots \cap \mathcal{C}_n$. Hence, the iterative scheme (21) solves the feasibility problem (17). Figure 1 shows a schematic representation of the given standard alternating projection algorithm for the case of two constraint sets $\mathcal{C}_1$ and $\mathcal{C}_2$.

For the case of two disjoint constraint sets $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e., if $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, the odd subsequence X_{2i-1} converges to the matrix X^* in the set C_1 that is nearest the set C_2. Hence, for the case of disjoint sets the iterative scheme (21) converges to the solution of the

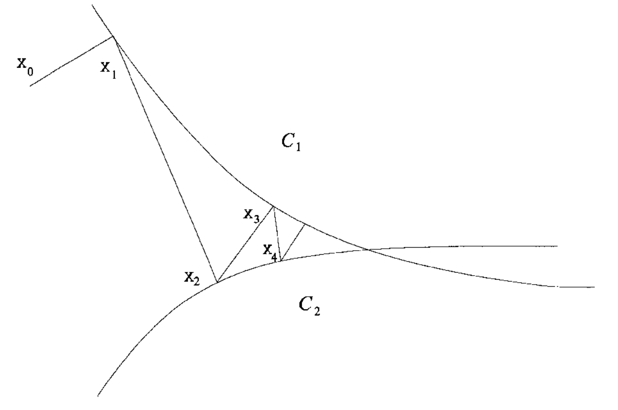

Fig. 1 Classic alternating projection scheme for feasible constraints C_1 and C_2.

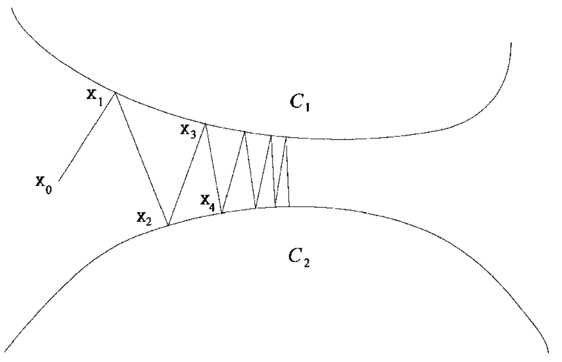

Fig. 2 Classic alternating projection scheme for unfeasible constraints $\mathcal{C}_1$ and $\mathcal{C}_2$.

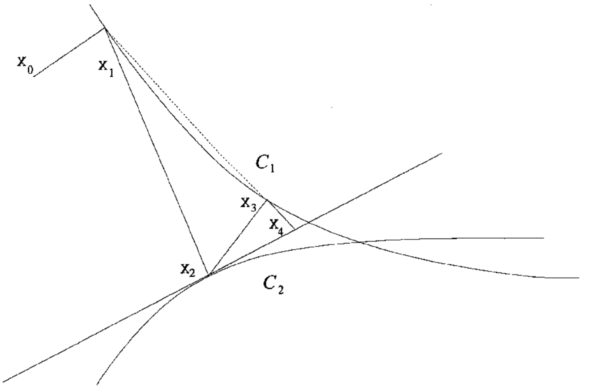

Fig. 3 Directional alternating projection scheme.

unfeasible optimization problem (19). Figure 2 shows the behavior of the algorithm (21) in this case.

B. Optimal Alternating Projection Algorithm

The following modified alternating projection[20] algorithm provides a solution to the optimization problem (18). Consider the matrix X_0 in $\mathcal{M}$ and define the following sequence of matrices:

$$
\begin{aligned}
X_1 &= \mathcal{P}_{\mathcal{C}_1} X_0, & Z_1 &= X_1 - X_0 \\
X_2 &= \mathcal{P}_{\mathcal{C}_2} X_1, & Z_2 &= X_2 - X_1 \\
&\vdots & &\vdots \\
X_n &= \mathcal{P}_{\mathcal{C}_n} X_{n-1}, & Z_n &= X_n - X_{n-1} \qquad (22)\\
X_{n+1} &= \mathcal{P}_{\mathcal{C}_1}(X_n - Z_1), & Z_{n+1} &= X_{n+1} - (X_n - Z_1) \\
X_{n+2} &= \mathcal{P}_{\mathcal{C}_2}(X_{n+1} - Z_2), & Z_{n+2} &= X_{n+2} - (X_{n+1} - Z_2) \\
&\vdots & &\vdots
\end{aligned}
$$

Hence, in this modified projection algorithm, in each step j, the increment Z_{j-n} is removed before projecting onto the corresponding convex set. This forces the algorithm to converge to the solution X^* of the optimization problem (18).

C. Directional Alternating Projection Algorithm

An algorithm that further exploits the geometry of the matrix design problem to enhance convergence is presented next. This algorithm uses the tangent plane onto a constraint set to solve the matrix feasibility problem (17). Let X_0 be any given matrix in $\mathcal{M}$. Consider the following algorithm for the case of the two constraint sets $\mathcal{C}_1$ and $\mathcal{C}_2$:

$$
\begin{aligned}
&X_1 = \mathcal{P}_{\mathcal{C}_1} X_0, \qquad X_2 = \mathcal{P}_{\mathcal{C}_2} X_1, \qquad X_3 = \mathcal{P}_{\mathcal{C}_1} X_2 \\
&X_4 = X_1 + \lambda_1 (X_3 - X_1), \qquad \lambda_1 = \frac{\|X_1 - X_2\|^2}{\langle X_1 - X_3, X_1 - X_2 \rangle} \\
&X_5 = \mathcal{P}_{\mathcal{C}_1} X_4, \qquad X_6 = \mathcal{P}_{\mathcal{C}_2} X_5, \qquad X_7 = \mathcal{P}_{\mathcal{C}_1} X_6 \qquad (23) \\
&X_8 = X_5 + \lambda_2 (X_7 - X_5), \qquad \lambda_2 = \frac{\|X_5 - X_6\|^2}{\langle X_5 - X_7, X_5 - X_6 \rangle} \\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}
$$

Then the sequence of matrices X_i converges to the intersection $\mathcal{C}_1 \cap \mathcal{C}_2$. Hence, the iterative scheme (23) solves the feasibility problem (17). This algorithm provides an improved convergence rate compared to the standard alternating projection algorithm (21) (see Ref. 20). Figure 3 shows a schematic representation of the directional alternating projection algorithm for the case of two constraint sets $\mathcal{C}_1$ and $\mathcal{C}_2$. Note that in each cycle, there are two projections points (X_1 and X_3) on the same set $\mathcal{C}_1$, and these two projection points are used to derive a direction for the next point X_4.

IV. Alternating Projections for Damage Detection

To apply and efficiently implement the alternating projection methods to the solution of damage detection problems the expressions for the projections onto the constraint sets $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ defined in Sec. II are required. In this section, analytical expressions for these projections are derived.

The projection on the positive definiteness constraint set $\mathcal{C}_1$ is obtained via an eigenvalue–eigenvector decomposition $L\Lambda L^T$ of the matrix K_d and the replacement of the negative eigenvalues by zero.[21] Hence, this projection is given by

$$K_d^* = \mathcal{P}_{\mathcal{C}_1}(K_d) = L\Lambda_+ L^T \tag{24}$$

where Λ_+ is the diagonal matrix obtained by replacing the negative diagonal elements in Λ by zero and L is the eigenvector matrix. Efficient symmetric matrix eigensolvers can be used to compute this projection.

The projection on the symmetry constraint set $\mathcal{C}_2$ is given by

$$K_d^* = \mathcal{P}_{\mathcal{C}_2}(K_d) = \frac{K_d + K_d^T}{2} \tag{25}$$

Hence, K_d^* is the symmetric part of K_d.

The sparsity pattern of the K_d matrix is obtained by preserving the zero/nonzero pattern of the analytical stiffness matrix K. The projection of a nonsparse matrix K_d onto the K-sparse constraint set C_3 is obtained by preserving the elements of K_d that correspond to the nonzero pattern of K and zeroing the elements that correspond to the zero pattern of K. Let us denote by S_K the matrix that consists of zeros and ones and has the sparsity pattern of K. Then the projection K_d^* of a matrix K_d onto the set C_3 can be obtained as

$$K_d^* = \mathcal{P}_{C_3}(K_d) = K_d \circ S_K \tag{26}$$

The projection onto the eigenequation constraint set $\mathcal{C}_4$ is obtained by finding the minimum distance solution of the eigenequation (4) with respect to the distance $\|K_d - K\|$. To this end, notice that subtracting the term KV_d from both sides of the eigenequation (4) provides

$$(K_d - K)V_d = MV_d\Omega_d^2 - KV_d \tag{27}$$

The minimum norm solution with respect to $K_d - K$ is obtained via the Moore–Penrose generalized inverse V_d^+ of V_d as follows:

$$K_d^* - K = \left(MV_d\Omega_d^2 - KV_d\right)V_d^+ \tag{28}$$

Hence, the projection K_d^* of a matrix K_d onto the set $\mathcal{C}_4$ is

$$K_d^* = \mathcal{P}_{\mathcal{C}_4}(K_d) = \left(MV_d\Omega_d^2 - K_dV_d\right)V_d^+ + K_d \tag{29}$$

Notice that the generalized inverse V_d^+ of the eigenvector matrix V_d needs to be computed only once at the beginning of the alternating projection algorithm.

To obtain the projection onto the masking constraint set $\mathcal{C}_3'$, a masking matrix I_m is constructed in each iteration of the alternating projection algorithm. This matrix consist of ones at the location where a significant change has occurred in the stiffness matrix and

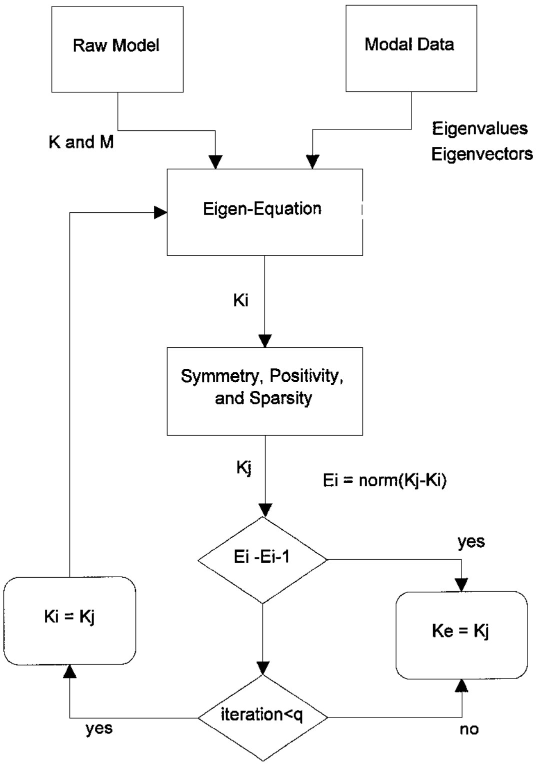

Fig. 4 Alternating projections for damage detection flowchart.

zeros at the location where no significant change has been detected, e.g., less that 10% of the maximum change. Obviously, I_m is a symmetric K-sparse matrix. Hence, the projection onto the masking constraint set $\mathcal{C}_3'$ is obtained by

$$K_d^* = \mathcal{P}_{\mathcal{C}_3'}(K_d) = K + (K_d - K) \circ I_m$$

Figure 4 depicts the flowchart of the alternating projection scheme for the damage detection problem.

V. Numerical Examples

A. Cantilever Beam Model

Consider the analytical FEM of a four-DOF cantilevered beam with mass matrix

$$M = 10^{-4} \times \begin{bmatrix} 1.5695 & 0 & 0 & 0 \\ 0 & 1.5695 & 0 & 0 \\ 0 & 0 & 1.5695 & 0 \\ 0 & 0 & 0 & 0.7848 \end{bmatrix}$$

and stiffness matrix

$$K = 10^{8} \times \begin{bmatrix} 5.5814 & -2.7907 & 0 & 0 \\ -2.7907 & 5.5814 & -2.7907 & 0 \\ 0 & -2.7907 & 5.5814 & -2.7907 \\ 0 & 0 & -2.7907 & 2.7907 \end{bmatrix}$$

To validate and evaluate the performance of the proposed damage detection algorithms, simulated damage was introduced at the third element of this model by decreasing the corresponding stiffness by 25%. We assume that three eigenfrequencies and mode shapes of the damaged model are available to detect this simulated damage. The corresponding eigenfrequency matrix Ω_d and the eigenvector matrix V_d are

$$\Omega = \begin{bmatrix} 4.8184 \times 10^5 & 0 & 0 \\ 0 & 1.2325 \times 10^6 & 0 \\ 0 & 0 & 2.200 \times 10^6 \end{bmatrix}$$

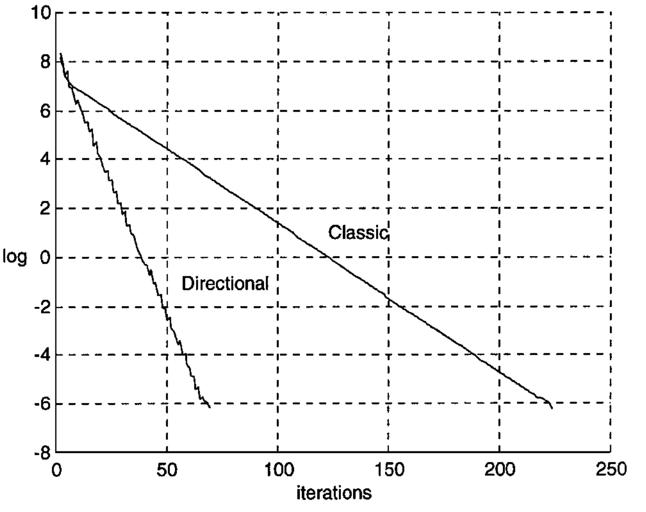

Fig. 5 Alternating projection iterations for the cantilever beam model.

and

$$V_d = \begin{bmatrix} 0.1986 & -0.5415 & -0.7080 \\ 0.3711 & -0.6204 & 0.5115 \\ 0.6194 & 0.2820 & 0.1654 \\ 0.6627 & 0.4923 & -0.4580 \end{bmatrix}$$

For comparison purposes, both the classic and the directional alternating projection algorithms (21) and (23) are examined for damage detection. In this noise-free problem the damaged structural stiffness constraints (10) have a nonempty intersection providing a unique solution to the optimization problem Eqs. (11) and (12). The predicted damaged stiffness matrix from the alternating projection algorithms is

$$K_d^* = 10^8 \times \begin{bmatrix} 5.5814 & -2.7907 & 0.0000 & 0.0000 \\ -2.7907 & 4.1860 & -1.3953 & 0.0000 \\ 0.0000 & -1.3953 & 4.1860 & -2.7907 \\ 0.0000 & 0.0000 & -2.7907 & 2.7907 \end{bmatrix}$$

which corresponds to the exact simulated damage stiffness matrix. That is, the alternating projection approach provides the exact answer for this simulated example. Both the standard alternating projection and the directional alternating projection algorithms converged to the given solution. Figure 5 shows the logarithm of the error of the iterative scheme vs the iteration number for these algorithms. Notice that the directional alternating projection method requires approximately one-third of the number of iteration that the standard alternating projection requires.

For comparison, an alternative damage detection approach that utilizes the Baruch–Bar Itzhack formula (7) in an iterative fashion is also used to detect the same simulated damage. This technique, proposed in Refs. 9 and 22, predicts the same exact simulated damage. In addition, a new iterative Baruch–Bar Itzhack approach that utilizes directional information for the projection (7), as proposed in algorithm (23), is applied to solve this problem. This directional iterative Baruch–Bar Itzhack approach has the advantage of converging faster to the solution of the simulated damage detection problem. A comparison of the convergence of these iterative schemes is shown in Fig. 6. It is noted that the utilization of the directional information significantly enhances the convergence rate.

B. NASA Eight-Bay Truss Damage Detection Test Bed

The NASA eight-bay truss structure is a cantilevered eight-bay structure designed at the NASA Langley Research Center to be used as an experimental test bed to collect high-quality experimental modal data for damage detection testing and validation purposes.[23] The structural model is a 96-DOF FEM model developed using rod elements and concentrated masses, as shown in Fig. 7. The truss has been instrumented with 96 accelerometers to measure the three

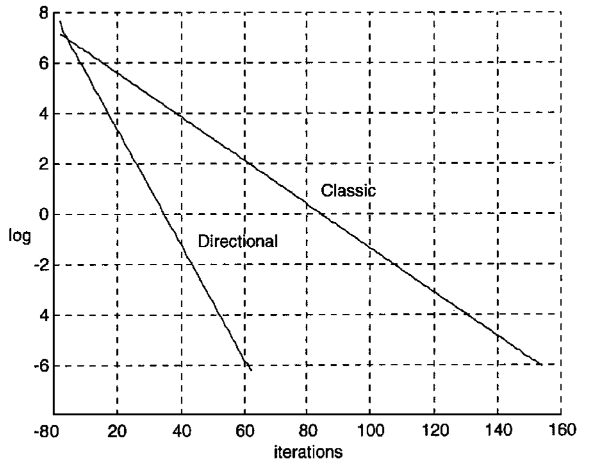

Fig. 6 Baruch–Bar Itzhack iterations for the cantilever beam model.

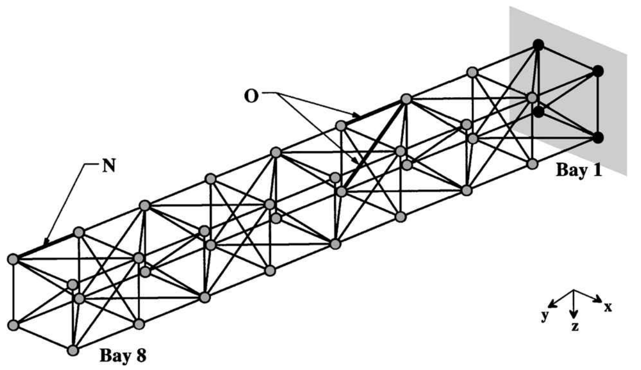

Fig. 7 NASA eight-bay experimental test bed and damage cases.

translational DOFs at each node. Damage has been inflicted by the removal of truss members at prespecified locations. Experimental modal data have been extracted for healthy structure and for different damage case scenarios that correspond to the removal of different truss members. These data are used in this study to validate the proposed alternating projection algorithms for damage detection.

We use the first five modes of vibration to solve the damage detection problem for different damage case scenarios. Before the implementation of the alternating projection algorithms, the finite element model was refined using available healthy modal data. The alternating projection algorithm of Sec. IV without masking was utilized for this model refinement.

Damage cases denoted as cases A, C, G, H, I, J, L, M, N, and O are examined. Two representative cases are discussed next. In all of these cases, 50 iterations of the alternating projection algorithm (21) were utilized to solve the noisy damage detection infeasible optimization problem (13–15).

The damage case N corresponds to the removal of a truss member at the free end of the truss (bay eight); hence, this is a low-strain damage case. DOFs 2 and 14 in the FEM model of the structure are affected. The results of the alternating projection damage detection scheme are shown in Figs. 8 and 9. Figure 8 shows a mesh plot of the absolute values of the perturbed stiffness matrix $\Delta K = K - K_d$ that is attributed to damage. Figure 9 shows the normalized damage indicator vector for each DOF for the same case. The damage at DOFs 2 and 14 is clearly represented, and the remaining residual peaks can be attributed to experimental data measurement errors.

The damage case O corresponds to the removal of two truss members near the base of the truss (bay three). DOFs 62, 68, 69, 74, and 75 are affected. Figure 10 shows the mesh plot of ΔK, and Fig. 11 shows the damage indicator vector. Damage information again can be extracted from the peaks values in these plots. The peak that corresponds to DOF 50 is attributed to a faulty sensor at this location for this experiment (private communication from T. Kashangaki).

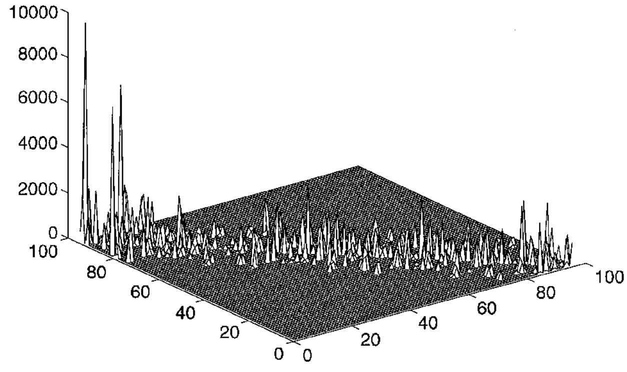

Fig. 8 Mesh plot of ΔK for the NASA eight-bay case N.

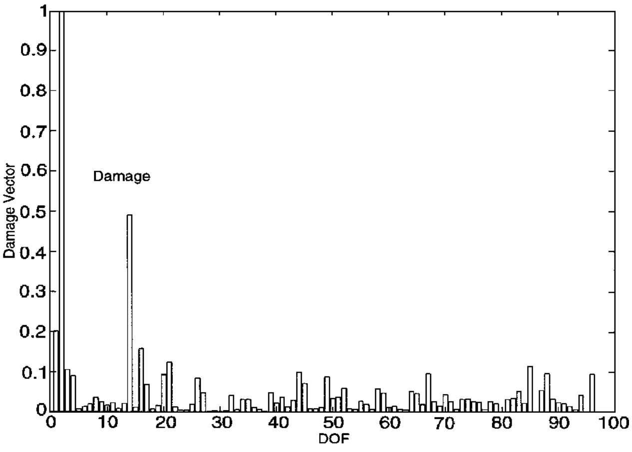

Fig. 9 Damage indicator vector for the NASA eight-bay case N.

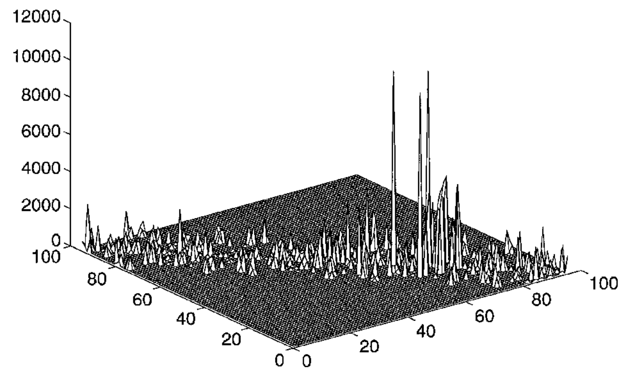

Fig. 10 Mesh plot of ΔK for the NASA eight-bay case O.

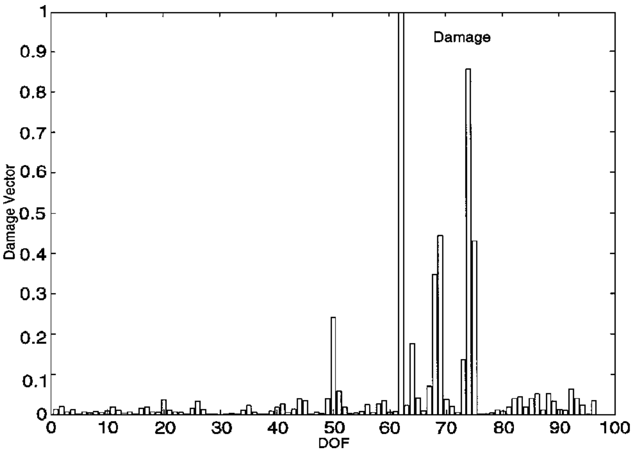

Fig. 11 Damage indicator vector for the NASA eight-bay case O.

Table 1 Damage extent estimates

	MRPT		Alternating projections method	
Case	Damage extent, lb/in. (N/m)	%Error	Damage extent, lb/in. (N/m)	%Error
A	11,896 (2.08351×10^6)	8.77	11,876 (2.08000×10^6)	8.93
C	11,901 (2.08438×10^6)	8.73	9,289 (1.62691×10^6)	19.82
G	10,859 (1.90188×10^6)	16.70	10,425 (1.82587×10^6)	20.05
H	12,601 (2.20698×10^6)	3.37	12,957 (2.26933×10^6)	0.64
I	7,568 (1.32549×10^6)	16.00	7,272 (1.27364×10^6)	19.32
J	12,015 (2.10435×10^6)	7.86	12,538 (2.19595×10^6)	3.85
L	8,421 (1.47488×10^6)	35.40	11,191 (1.96003×10^6)	14.18
M	12,009 (2.10330×10^6)	7.91	11,712 (2.05128×10^6)	10.18
N	10,037 (1.75792×10^6)	23.00	8,828 (1.54617×10^6)	32.30
O	12,347 (2.16250×10^6)	5.31	12,344 (2.16197×10^6)	5.34

Notice that the mesh plots (Figs. 8 and 10) have been rotated by 90 deg to depict the damage results more clearly. Table 1 presents a summary of the damage detection results for all cases that have been examined. The extent of the damage is estimated based on the calculated stiffness reduction at the damaged locations. For comparison, the corresponding damage detection results from the MRPT are also presented. However, the MRPT results include a filtering treatment after the damage detection step that improves the accuracy of the damage extent estimate.[12] Notice that MRPT provides a fully populated damaged stiffness matrix that does not satisfy the positive definiteness and connectivity constraints of the physical problem.

VI. Conclusions

The use of alternating projection methods for damage detection in structures is examined. These methods provide iterative schemes that enforce symmetry, sparsity, positive definiteness, and eigenequation constraints for the damage stiffness matrix to improve the damage detection estimates using modal test data. The proposed approach can accommodate noisy modal data by obtaining the damage stiffness matrix that approximates the noisy eigenproblem and satisfies the remaining physical constraints. Application of the alternating projection methods to detect damage using simulated and experimental data shows satisfactory performance. Both damage location and extent information are extracted for the NASA eight-bay experimental damage detection test bed using noisy modal data. Exact damage location is predicted for 10 damage test cases, although the damage extent error ranges from 0.5 to 30% depending on the case.

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References

[1]Baruch, M., and Bar Itzhack, I. Y., "Optimum Weighted Orthogonalization of Measured Modes," *AIAA Journal*, Vol. 16, No. 4, 1978, pp. 346–351.

[2]Ibrahim, S. R., and Saafan, A. A., "Correlation of Analysis and Test in Modeling of Structures, Assessment and Review," *Proceedings of the Fifth International Modal Analysis Conference* (London), Society for Experimental Mechanics, Bethel, CT, 1987, pp. 1651–1660.

[3]Zimmerman, D., and Smith, S. W., "Model Refinement and Damage Location for Intelligent Structures," *Intelligent Structural Systems*, edited by H. S. Tzou and G. L. Anderson, Kluwer Academic, Norwell, MA, 1992, pp. 403–452.

[4]Mottershead, J. E., and Friswell, M. I., "Model Updating in Structural Dynamics: A Survey," *Journal of Sound and Vibration*, Vol. 167, No. 2, 1993, pp. 347–375.

[5]Doebling, S. W., Farrar, C. R., Prime, M. B., and Sheritz, D. W., "Damage Identification and Health Monitoring of Structural and Mechanical Systems from Changes in Their Vibration Characteristics: A Literature Review," Los Alamos National Lab., Rept. LA-13070-MS, Los Alamos, NM, May 1996.

[6]Kabe, A., "Stiffness Matrix Adjustment Using Mode Data," *AIAA Journal*, Vol. 23, No. 9, 1985, pp. 1431–1436.

[7]Kammer, D., "Optimum Approximation for Residual Stiffness in Linear System Identification," *AIAA Journal*, Vol. 26, No. 1, 1988, pp. 104–112.

[8]Smith, S., and Beattie, C., "Simultaneous Expansion and Orthogonalization of Measured Modes for Structural Identification," *Proceedings of the AIAA Dynamic Specialist Conference* (Long Beach, CA), AIAA, Washington, DC, 1993, pp. 261–270.

[9]Smith, S., "Iterative Use of Direct Matrix Updates: Connectivity and Convergence," *Proceedings of the AIAA 33rd Structures, Structural Dynamics, and Materials Conference* (Dallas, TX), AIAA, Washington, DC, 1992, pp. 1797–1806.

[10]Zimmerman, D., and Kaouk, M., "Eigenstructure Assignment Approach for Structural Damage Detection," *AIAA Journal*, Vol. 30, No. 7, 1992, pp. 1848–1855.

[11]Zimmerman, D., and Kaouk, M., "Structural Damage Detection Using a Minimum Rank Update Theory," *Journal of Vibration and Acoustics*, Vol. 116, 1994, pp. 222–231.

[12]Zimmerman, D., Kaouk, M., and Simmermacher, T., "Structural Health Monitoring Using Vibration Measurements and Engineering Insight," *Transactions of the ASME, Special 50th Anniversary Design Issue*, Vol. 117, 1995, pp. 214–221.

[13]Combettes, P. L., "The Foundations of Set Theoretic Estimation," *IEEE Proceedings*, Vol. 81, No. 2, 1993, pp. 182–208.

[14]Combettes, P. L., and Trussell, H., "Method of Successive Projections for Finding a Common Point of Sets in Metric Spaces," *Journal of Optimization Theory and Applications*, Vol. 67, No. 3, 1990, pp. 487–507.

[15]Grigoriadis, K., and Skelton, R., "Alternating Convex Projection Methods for Covariance Control Design," *International Journal of Control*, Vol. 60, No. 6, 1994, pp. 1083–1106.

[16]Grigoriadis, K., and Skelton, R., "Low-order Control Design for LMI Problems Using Alternating Projection Methods," *Automatica*, Vol. 32, No. 8, 1996, pp. 1117–1125.

[17]Luenberger, D., *Optimization by Vector Space Methods*, Wiley, New York, 1968, Chap. 3.

[18]Cheney, W., and Goldstein, A., "Proximity Maps for Convex Sets," *Proceedings of the American Mathematical and Society*, Vol. 12, No. 3, 1959, pp. 448–450.

[19]Bolye, S., and Dykstra, R., "A Method for Finding Projections onto the Intersection of Convex Sets in Hilbert Space," *Advances in Order Restricted Statistical Inference*, Vol. 37, Lecture Notes in Statistics, Springer–Verlag, Berlin, 1986, pp. 28–47.

[20]Gubin, L., Polyak, B., and Raik, E., "The Method of Projections for Finding the Common Point of Convex Sets," *USSR Computational Mathematics and Mathematical Physics*, Vol. 7, 1967, pp. 1–24.

[21]Higham, N. J., "Computing the Nearest Symmetric Positive-Semidefinite Matrix," *Linear Algebra and its Applications*, Vol. 103, May 1988, pp. 103–118.

[22]Kaouk, M., and Zimmerman, D., "Structural Damage Assessment Using a Generalized Minimum Rank Perturbation Theory," *AIAA Journal*, Vol. 32, No. 4, 1994, pp. 836–842.

[23]Kashangaki, T., "Ground Vibration Tests of a High Fidelity Truss for Verification of on Orbit Damage Location Techniques," NASA TM-107626, 1992.

A. Berman
Associate Editor